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The equations of the magnetosphere

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Abstract
The use of equations, and mathematical modelling in magnetospheric and space physics is reviewed. First, the basic equations are discussed. Then, kinetic and fluid theory is treated. The role of approximations and the applicability of the theories in practice are emphasised.

1 Thoughts on equations

The topic of equations in magnetospheric science is vast. It involves the fundamental equations of electromagnetics, Newton's laws for particle motion and the theory of relativity, which are crucial not only to the understanding of our field, but indeed most if not all of physics. On the other end of the scale, we have equations that are used by researchers to explain a particular observation, and that cannot be generalised to other situations. In between we find equations that apply to a particular problem, such as the current–voltage relationship of the aurora, that while not fundamental nevertheless are often used by many scientists in the field.

The vast majority of the magnetosphere, at least in terms of volume, is a collisionless plasma, and it can be described by the equations governing collisionless plasma physics. However, the interface toward the ionosphere at the magnetosphere's inner boundary is not collisionless at all. In fact, it is through collisions that we can see the aurora, the only magnetospheric phenomenon that is observable with the naked eye and without scientific instrumentation.

Speaking of equations, it may also be worthwhile to reflect upon why we use them and how we best can accomplish what we want with, or perhaps without, the use of equations. Biot–Savart's law, which in modern textbooks is written as (e.g. D. K. Cheng, 1989)

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l'} \times \vec{a}_R}{R^2} \tag{1}$$

was published by Biot & Savart (1820) in an article, about one page long, which contained no equations and no figures. In this case, a single equation combined with a small figure, defining the quantities involved would more efficiently convey the relationship between the current and the magnetic field. Thus, we can talk about nature in the language of mathematics, which is understood also by those who are unfamiliar with the language in which the original publication is written. This being said, one must also acknowledge that the mathematical language sometimes is not always readily comprehended even by colleagues in the field and that a physical understanding often may be easier to convey by other means, particularly when the study itself involves lengthy derivations of equations.

Furthermore, describing our findings mathematically allows for quantitative predictions. The ability to make predictions is necessary in developing science-based technical applications, but also to understand science itself when we move beyond simple relationships between a small number of variables. For example, the plasma waves that appear in the various parts of the magnetosphere are derived mathematically, and we would hardly be able to understand the physics behind them without that mathematical description. When analysing satellite data it is by comparison to theoretical predictions of wavelengths, frequencies and directions of propagation that we can identify wave modes and, in turn, generation mechanisms and energy flows. Thus, the mathematical description is more than a language used for efficiency in lieu of other languages. It is an integral part of modern magnetospheric physics, and we cannot do without it.

In spite of the above example of an equationless publication from 1820, the need for quantitative predictions was already realised at the time and the mathematical treatment of the natural sciences was emerging as can be seen by the example of Poisson's equation, which is of great importance in our field (Poisson, 1813). The field of numerical simulations is entirely based on the numerical treatment of equations, and experiments can be conducted completely in the computer with no connection to reality. Once the equations that are used have been established, when their limitations are known, and how initial and boundary conditions are put in relation to observations, these computer experiments can be conducted much like laboratory experiments. It is then possible to publish scientific papers that, although they rely completely on the mathematical description, contain no equations at all (e.g. Gunell et al., 2007, my own paper – not to embarrass anybody else). Thus, what existed first as a purely theoretical field of study has created a new field that is essentially experimental.

Computer simulations can be very successful in advancing our understanding of magnetospheric physics. In addition to the purely numerical challenges of the field, it is imperative to know the limitations of the numerical models used, to establish the validity of the models to the problem under study, and to confirm as much as possible that the numerical results agree with observations. There is not always a clear answer to the question of which model is the most suitable to a particular problem. A model may describe some aspects of a phenomenon well, while failing to describe others, and then the choice of model depends not only on the physics of the the object of study, but also on the question one endeavours to answer.

The aim of this paper is to review, briefly, some of the techniques in common use in magnetospheric and space physics; to shed some light on the regimes of applicability of these models, and to provide a few examples of how these methods are used today. For a complete treatment with detailed derivations of the equations one has to turn to textbooks, for example the book by Krall & Trivelpiece (1973), which has been a useful source of information to the writer of these pages. I have endeavoured to provide examples of mathematical modelling of various phenomena from the parts of magnetospheric physics with which I am familiar. The list is not exhaustive nor restricted to Earth's magnetosphere, since the underlying principles that govern the behaviour of our planet are shared with other solar system objects. In other words, in this chapter, the author goes on and on about stuff. The examples brought to mention here do not cover the complete history of the field, and it is very likely that I have forgotten important works. Hopefully, those that I have remembered will be able to illustrate the successes and challenges of mathematical modelling in magnetospheric physics today.

2 Basic equations

In magnetospheric physics, like everywhere else, the electric and magnetic fields can be found as solutions to Maxwell's equations.

$$
\begin{cases}
\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & (2)\\[2mm]
\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \dfrac{\partial \vec{E}}{\partial t} & (3)\\[2mm]
\nabla \cdot \vec{E} = \dfrac{\rho}{\epsilon_0} & (4)\\[2mm]
\nabla \cdot \vec{B} = 0 & (5)
\end{cases}
$$

The notation is explained in table 1. In a plasma, the sources, ρ and $\vec{J}$, to the electromagnetic fields are given by the particle positions and velocities. For a complete description we thus need to model the motion of all charged particles. We may define a function that specifies the positions and velocities for the $\bar{N}_\alpha$ particles of species α

Table 1. The quantities represented by the symbols used in this chapter.

$\vec{E}$	Electric field	μ_0	Permeability of free space
$\vec{B}$	Magnetic flux density	ϵ_0	Permittivity of free space
$\vec{J}$	Current density	c_0	Speed of light in vacuum
ρ	Charge density	α	Particle species
ρ_{m}	Mass density	$\vec{x}$	Particle position
σ	Conductivity	$\vec{v}$	Particle velocity
f	Distribution function	λ_{D}	Debye length
n	Plasma density	ω	Angular frequency
n_{e}	Electron density	$\overline{\overline{P}}$	Pressure tensor
n_{i}	Ion density	e	elementary charge
k	Wave number	$N_\alpha(\vec{x},\vec{v},t)$	Klimontovich-Dupree distribution function
ν	Collision frequency	$\bar{N}_\alpha$	Total number of particles of species α

(Klimontovich, 1958; Dupree, 1963).

$$N_\alpha(\vec{x},\vec{v},t) = \sum_{1\le j\le \bar{N}_\alpha} \delta\left(\vec{x}-\vec{x}_j(t)\right)\delta\left(\vec{v}-\vec{v}_j(t)\right) \tag{6}$$

Integrating Eq. (6) over all phase space we obtain the total number of particle of species α:

$$\bar{N}_\alpha = \int N_\alpha(\vec{x},\vec{v},t)d\vec{x}d\vec{v}. \tag{7}$$

The charge density in Eq. (4) and the current density in Eq. (3) are found by integration

$$\rho = \sum_\alpha q_\alpha \int N_\alpha(\vec{x},\vec{v},t)d\vec{v} \tag{8}$$

$$\vec{J} = \sum_\alpha q_\alpha \int \vec{v}N_\alpha(\vec{x},\vec{v},t)d\vec{v} \tag{9}$$

Assuming that there is no particle production nor any losses and that only electric and magnetic forces act on the particles, the equations of motion for particle j are

$$\frac{d\vec{x}_j}{dt} = \vec{v}_j \tag{10}$$

$$\frac{d\vec{v}_j}{dt} = \frac{q_j}{m_j}\left(\vec{E}+\vec{v}_j\times\vec{B}\right) \tag{11}$$

Due to the conservation of particles in phase space $dN_\alpha(\vec{x},\vec{v},t)/dt = 0$ which, using the equations of motion becomes

$$\frac{\partial N_\alpha(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\frac{\partial N_\alpha(\vec{x},\vec{v},t)}{\partial \vec{x}} + \frac{q_\alpha}{m_\alpha}\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\frac{\partial N_\alpha(\vec{x},\vec{v},t)}{\partial \vec{v}} = 0 \tag{12}$$

The fields in Eq. (12) are the microscopic fields that each particle feels from all the other particles. For convenience it has not been explicitly stated in Eq. (12) that when evaluating the fields at the particle position, the contributions from the particle itself must be removed (see Dupree, 1963). Equation (12) looks conspicuously like the Vlasov equation, which we shall meet in Section 3, but unlike that equation, Eq. (12) includes the interaction between individual particles, and can therefore describe fluctuations due to particle discreteness that are otherwise ignored in kinetic theory. Because this description requires modelling the motion of all particles it is not practical beyond very small systems. Nevertheless, there are practical applications where the effects of particle discreteness are important. Scattering of electromagnetic radiation is a single particle effect, and incoherent scattering radars (Gordon, 1958) rely on it, because without the discrete particles there would be no scattering centres.

Thermal fluctuations in the plasma are caused by the motion of individual particles, which gives rise to collective wave modes. Power spectra of these thermal fluctuations can be computed through superposition of dressed test particles (Rostoker, 1964a,b). In the dressed test particle model, each particle is treated as a Debye-shielded, *dressed*, test particle; the waves it generates as it moves through the plasma are computed, and the contributions from all such test particles are added to yield the final spectrum. A plasma is often defined as an ionised gas that exhibits collective properties. In the dressed test particle method, the particles are – one by one – taken out of the plasma, and its response to their presence is examined. In incoherent scattering radars, it is the width of the ion fluctuation spectrum that determines the width of the scattered power spectrum, and not as one naively could believe, the thermal spread of the electron distribution (Bowles, 1958; Fejer, 1960; Hagfors, 1961; Rosenbluth & Rostoker, 1962). This shows the importance of always remembering that the kinetic and fluid descriptions are approximations, and that there are phenomena that can be understood only by going back to the most basic equations.

3 Kinetic theory

Kinetic theory is a statistical description of the plasma, where one considers the distribution function $f(\vec{x}, \vec{v}, t)$ which is defined so that the number of particles in an element $d\vec{x}d\vec{v}$ of the six-dimensional phase space at time t is

$$f(\vec{x}, \vec{v}, t)d\vec{x}d\vec{v}.$$

As there almost always are more than one particle species in the plasma, we define separate distribution functions f_α for each species. Under the influence of electromagnetic forces, the distribution function satisfies the Vlasov equation (Vlasov, 1968, *translated from* (Vlasov, 1938))

$$\frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \frac{\partial f_\alpha}{\partial \vec{x}} + \frac{q_\alpha}{m_\alpha}\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \frac{\partial f_\alpha}{\partial \vec{v}} = 0. \tag{13}$$

Vlasov was first to use this equation in plasma physics, but equations of this form were known a full century earlier, when Liouville (1838) examined purely mathematical equation properties. In gas dynamics, the Boltzmann equation (Boltzmann, 1896) is an equation of the same kind which includes a collision term, and Jeans (1915) used an equation of this form to study the motion of stars. Henon (1982) argued that, because of this history, a better name for the equation would be "collisionless Boltzmann equation", but the name for Eq. (13) that stuck – at least in this field – is the *Vlasov equation*.

The Vlasov equation, Eq. (13), is the zeroth order kinetic equation describing the plasma, in which all particle to particle interactions have been neglected. By assuming each particle interacts directly with one other particle, the first order kinetic equation can be derived. By including interactions between each particle and two others, one obtains the second order kinetic equation, and so on (Krall & Trivelpiece, 1973). The condition that allows us to neglect binary interactions is that there are many particles in a Debye cube,

$$\frac{1}{n\lambda_{\mathrm{D}}^3} \ll 1. \tag{14}$$

This can be understood by considering two particles that occupy the same small volume within the Debye sphere, or cube. The motion of one of these particles will be more influenced by the many particles in the Debye sphere than by the only one other particle within the small volume. Thus, if Eq. (14) is satisfied collective effects dominates over single particle effects, and that is how we usually define a plasma. For practical purposes, this sets the lower limit to the length scales for which conclusions can be drawn from zeroth order kinetic theory to approximately the Debye length. For shorter length scales, the word plasma may no longer be the most accurate description. For time scales, the Vlasov equation is valid for times shorter than typical collision times.

In space, collision frequencies are often very low, and the Vlasov equation and Maxwell's equations together provide an excellent description of the plasma. When the collision times are longer than other relevant time scales, for example the plasma period and the electron and ion cyclotron periods, the distributions do not always thermalise into Maxwellian distributions, and space plasmas often have non-Maxwellian distributions, exhibiting supra-thermal tails that can be modelled using for example Kappa distributions (Pierrard & Lazar, 2010) or simple pole expansions (Löfgren & Gunell, 1997; Gunell & Skiff, 2001, 2002). One application of kinetic theory is to compute dispersion relations for waves. In the electrostatic case, Eqs. (13) and (4) are linearised and Fourier transformed, and a relationship between ω and k can be found. A consequence of linearising is that the results are only accurate for small amplitudes. For ion time scale waves in plasmas with non-Maxwellian distributions, Skiff et al. (2002) found that kinetic modes, that is to say, modes not well described by fluid theory, become important.

Another way in which kinetic theory can be used is to perform computer simulations to find how the plasma develops with time, given specific initial and boundary conditions. The two major classes of kinetic simulation methods are Vlasov simulations and particle simulations. In Vlasov simulations phase space is discretised so that the distribution function is known at the nodes of a grid. With knowledge of the distribution function, the fields can be computed at the grid points. Then, with knowledge of the fields, the phase space fluxes are computed, the distribution function is updated and this processes is repeated over and over, advancing the distribution function in time. The methods used usually build on the splitting scheme (C. Z. Cheng & Knorr, 1976). In particle simulations, the distribution function is represented by a number of particles, often several orders of magnitude fewer than the number of particles in the real plasma. The charge and current densities are transferred to a grid, and the fields are calculated on that grid. Then the particles are moved under influence of these fields and the process is repeated (Birdsall & Langdon, 1991). Even though particle in cell simulations (PIC) are using particles, they are not including particle to particle interactions and should be seen as a method for solving the Vlasov equation. Numerical kinetic modelling is described in more detail in Chapter 38.

In recent years, Vlasov simulations have been used in magnetospheric physics for example to study electrostatic acceleration of auroral electrons in the upward (Gunell et al., 2013) and downward (Gunell et al., 2015) current regions, and large scale simulations of the magnetosphere have been performed of both the nightside (Palmroth et al., 2017) and dayside (Palmroth et al., 2018) regions. Those large scale simulations employed a hybrid scheme where only the ions were modelled kinetically, and the electrons are there as a mere neutralising fluid. Such hybrid schemes are necessary as one cannot achieve the spatial and temporal resolutions required to simulate both electrons and ions in a simulation that includes the whole magnetosphere. The same idea is often employed in hybrid particle simulations, where the ions are treated as particles and the electrons as a fluid, and such hybrid models have been used extensively to study planets and other solar system objects (for example Kallio & Janhunen, 2001; Müller et al., 2011). There are also implicit methods (Markidis et al., 2010, Chapter 35, this volume), where the electrons are included as particles, but the electron plasma period is not resolved. In all these methods some of the physics is lost. That is the price one has to pay for the ability to perform global simulations, and it is the responsibility of the modeller to make sure that what is lost is not important to the problem that is being addressed.

4 Fluid theory and magnetohydrodynamics

A set of fluid equations can be obtained by taking moments of the Vlasov equation, combining these with Maxwell's equations, and closing the system of equations with a suitable equation of state. Depending on the assumptions that are made, widely differing phenomena can be described. Dispersion relations for waves in plasmas, such as Langmuir waves and ion acoustic waves, are often derived in this way in textbooks.

One particular theory of some interest in magnetospheric physics is magnetohydrodynamics (MHD). Alfvén (1942) used this set of equations

$$\begin{cases} \nabla \times \vec{B} = \mu_0 \vec{J} & (15) \\ \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & (16) \\ \vec{J} = \sigma \left(\vec{E} + \vec{v} \times \vec{B} \right) & (17) \\ \rho_{\mathrm{m}} \dfrac{\partial \vec{v}}{\partial t} = \vec{J} \times \vec{B} - \nabla p & (18) \end{cases}$$

for a magnetised fluid, assuming the plasma to be incompressible and $\sigma = \infty$ to derive the "electromagnetic-hydrodynamic" wave that propagates along the background magnetic field with phase speed

$$v_{\mathrm{A}} = \frac{B_0}{\sqrt{\mu_0 \rho}}. \tag{19}$$

Now these waves are known as Alfvén waves and v_{A} the Alfvén speed. Eqs. (15)–(18) are known as the MHD equations, and when $\sigma = \infty$ is assumed we have ideal MHD. These equations predicted the Alfvén waves, which subsequently were observed in experiments with liquid metals (Lehnert, 1958) and in the magnetosphere (Cummings et al., 1969). While the use of ideal MHD in space physics relies on many simplifying assumptions, this treatment is able to predict phenomena that do exist and have been observed. It is important to consider what the limitations are. The approach of Alfvén (1942) was to assume a perfectly conducting incompressible fluid and examine the consequences. If we instead start with a kinetic description and derive the fluid equations by computing the moments of Eq. (13) – with a collision term on the right-hand side, making it a Botzmann equation – we may be able to determine when certain assumptions are valid. In a single-fluid model the momentum equation then becomes

$$\rho_{\mathrm{m}} \frac{\partial \vec{v}}{\partial t} + \rho_{\mathrm{m}} \left(\vec{v} \cdot \nabla \right) \vec{v} = \rho \vec{E} + \vec{J} \times \vec{B} - \nabla \cdot \overline{\overline{P}}, \tag{20}$$

where $\overline{\overline{P}}$ is the plasma pressure tensor. The generalised Ohm's law is obtained by multiplying the equations for the first moment by q_α / m_α for electrons and ions and adding the two equations to form

$$\begin{aligned} \frac{\partial \vec{J}}{\partial t} &+ \nabla \cdot \left(\vec{v}\vec{J} + \vec{J}\vec{v} - \vec{v}\vec{v}\rho \right) = \left(\frac{n_e e^2}{m_{\mathrm{e}}} + \frac{n_i e^2}{m_{\mathrm{i}}} \right) \vec{E} \\ &+ \left(\frac{e^2}{m_{\mathrm{e}}} + \frac{e^2}{m_{\mathrm{i}}} \right) \frac{\rho_{\mathrm{m}} \vec{v} \times \vec{B}}{m_{\mathrm{e}} + m_{\mathrm{i}}} - \left(\frac{e m_{\mathrm{i}}}{m_{\mathrm{e}}} - \frac{e m_{\mathrm{e}}}{m_{\mathrm{i}}} \right) \frac{\vec{J} \times \vec{B}}{m_{\mathrm{e}} + m_{\mathrm{i}}} \\ &- \frac{e}{m_{\mathrm{e}}} \nabla \cdot \left(\overline{\overline{P}}_{\mathrm{i}} \frac{m_{\mathrm{e}}}{m_{\mathrm{i}}} - \overline{\overline{P}}_{\mathrm{e}} \right) - \nu \vec{J}. \end{aligned} \tag{21}$$

For simplicity a plasma constituted of electrons and one singly charged ion species ($q_\alpha = e$) has been assumed, and the collision term has been approximated using the average collision frequency ν. We also need equations of continuity for the mass and charge densities:

$$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_{\mathrm{m}} \vec{v}) = 0 \tag{22}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0. \tag{23}$$

Equations (20)–(23) forms a set of single fluid equations, which in order to be solved need be closed by an equation of state, relating the pressure and density, for example $p \propto \rho_{\mathrm{m}}$ for an isothermal fluid; $p \propto \rho_{\mathrm{m}}^{\gamma}$, where $\gamma = C_p/C_V$ is the specific heat ratio, for an adiabatic fluid or $\nabla \cdot \vec{v} = 0$ for an incompressible fluid.

In going from Eqs. (15)–(18) to Eqs. (20)–(23) a number of approximations have been made. Observing that $m_{\mathrm{e}} \ll m_{\mathrm{i}}$ will simplify Eq. (21) somewhat. Quasi-neutrality will make $\rho = 0$, and if small perturbations around an equilibrium are considered the term $\nabla \cdot (\vec{v}\vec{J} + \vec{J}\vec{v} - \vec{v}\vec{v}\rho)$ in Eq. (21) can be neglected, since it is of second order. The term containing $\vec{J} \times \vec{B}$ in Eq. (21) is negligible in comparison to the term containing $\vec{v} \times \vec{B}$, if characteristic length scales over which the quantities involved change is long enough, because according to Eq. (16), $\vec{J}$ is proportional $\nabla \times \vec{B}$. If also the temporal changes are slow enough, the $\partial \vec{J}/\partial t$ term can be neglected. Similarly, the pressure gradient can be neglected for large length scales and for low pressure plasmas in strong magnetic fields.

In Eq. (18) the divergence of the pressure tensor has been approximated by a pressure gradient. The off-diagonal terms of $\overline{\overline{P}}$ may be neglected if the Reynolds number is large so that viscosity is unimportant. Furthermore, the use of a scalar pressure means that pressure isotropy has been assumed. In a collision-dominated plasma, isotropy is ensured, and even in collisionless plasmas this approximation is often used successfully. If the collision frequency is low, other processes on faster time scales act as effective collisions to isotropise the plasma. Thus, MHD, under the assumption of an isotropic pressure, can be applicable to large and slow scales, even though it cannot say anything about the processes on small and fast scales that are necessary to maintain that applicability.

By assuming quasi-neutrality (Schottky, 1924) the space charge density is taken to be zero, that is to say, $\rho = 0$, without placing the corresponding restriction on the divergence of the electric field. Thus, Eq. (4) is violated in this approximation, and we may very well have $\nabla \cdot \vec{E} \neq 0$. If we find $\vec{E}$ in quasi-neutral theory, Eq. (4) can be used to compute the charge density, ρ, that gave rise to that field. The plasma is not neural – only quasi-neutral. Even though this paragraph is in the section about fluid theory, quasi-neutrality is used in kinetic theory too. For example, Chiu & Schulz (1978) used a quasi-neutral kinetic model of an auroral field line to find that significant electric fields parallel to the magnetic field exist over distances of several Earth radii due to the magnetic mirror configuration. When does quasi-neutrality not apply? The electric field around a test particle that is placed in a plasma falls off on a typical spatial scale of a Debye length. However, while the spatial scale over which the plasma can sustain a deviation from quasi-neutrality is related to the Debye length, $1\,\lambda_D$ is not an upper limit to it. In electric double layers, space charge effects are generating a potential drop, and these structures can be “some tens of plasma Debye lengths” (Torvén & Andersson, 1979).

Global numerical modelling is discussed in Chapter 37. Here, let us briefly consider one example of a situation where considerations of the approximations made matter, namely magnetic reconnection. If the plasma truly obeyed the ideal MHD equations, the field lines would always be frozen to the plasma and reconnection would be impossible. Of course, the plasma is not an ideal MHD fluid, and field lines reconnect all the time. In resistive MHD, reconnection is possible in principle, but it has been found that it is necessary to include Hall effects to obtain realistic reconnection rates (Birn et al., 2001). Moreover, two-fluid effects have been shown to be important for a more detailed description of the physics involved (Yamada et al., 2010). Also pressure anisotropy and kinetic effects (Egedal et al., 2013) influence the reconnection process. At Jupiter's moon Ganymede (Chapter 35, this volume), Hall MHD has proved better than resistive MHD at predicting a configuration of field aligned currents that agree with observations of auroral emissions (Dorelli et al., 2015).

5 Test particle models

Both kinetic and fluid models, described in sections 3 and 4 respectively, are self-consistent. They account for both how the fields affect the particles and how the particles affect the fields. Test particle simulations is a class of simplified models that are not self-consistent. Instead the fields are prescribed, and the particle trajectories that result from those given fields are calculated by integrating the equation of motion with the Lorentz force acting on the particles. This can be useful in cases where the particles do not affect the fields to a significant extent. For example, in Earth's radiation belts that were discovered at the dawn of the space age (Van Allen et al., 1958) have been modelled in this way (Roederer, 1967, Chapter 21, this volume).

Another example of the use of test particle models is to study a minor species that does not affect the behaviour of the plasma. For example, charge-exchange X-rays are caused when highly charged ions (O_6^+, C_6^+, Ne_8^+ etc.), which constitute a small fraction of the solar wind, undergo charge-exchange collisions with neutrals in the exosphere of a planet. The X-ray emissions can be modelled by first using a self-consistent hybrid model of the interaction between the planet and the solar wind to find the electric and magnetic field. Then a test particle model can be used to compute the trajectories of the highly charged ions and the resulting emissions (Gunell et al., 2004, 2007).

The test particle simulation can be useful for specific purposes as shown by these examples, but not being self-consistent it remains an incomplete description of the plasma.

6 Summary

Now that we have reached the end of the last section before the concluding words, let us examine whether it is possible to draw a simple picture that makes sense of it all. An attempt at that is shown in Fig. 1, which illustrates of how the main classes of plasma theory described in this chapter are related to each other. With Maxwell's equations, Newton's laws of motion, and the Lorentz force we can model how all particles move and how the particles, in turn, affect the electric and magnetic fields. As following all particles is impractical in most cases, one can instead use a statistical model where the development of the distribution function is considered, and that is known as kinetic theory. By forming moments of the distribution function fluid theory is obtained. It does not end there. Combinations of both fluid and kinetic theory can be used in hybrid models, and the fields found in either fluid or kinetic theory can be used to compute particle trajectories in test particle simulations. Can we also make a figure that illustrates under what conditions the different theories should be used? This turns out to be much more difficult. When deciding on what model to use there are many choices to be made. Can the plasma be described by one or several fluids? Is the problem electrostatic or electromagnetic? How many dimensions are required in configuration space and in velocity space? It is not unusual that two different models can be used to study the same plasma, depending on what aspects of it are emphasised.

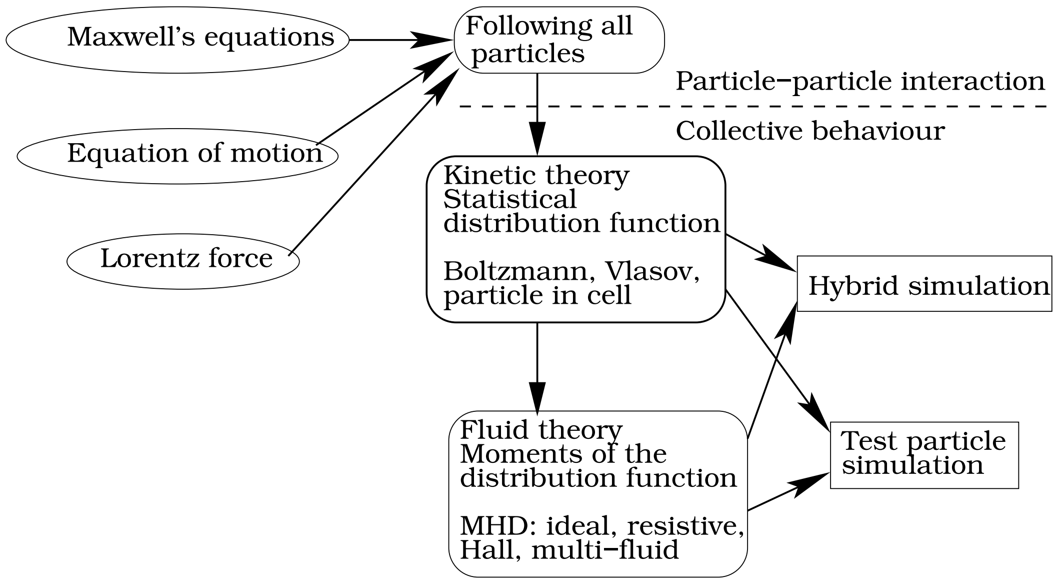

Figure 1. Schematic figure designed to illustrate the relationship between classes of plasma models in common use.

7 Conclusions

The equations of magnetospheric physics are much the same as those of electromagnetic theory, collisionless plasma physics, the kinetic theory of gases, and fluid dynamics. In any practical application of mathematics in space physics, approximations have to be made, and it is imperative to know the limitations of the models one intends to apply to a particular problem. Even when these limitations are known, assessing whether a model is applicable to a problem is no trivial task. If we, for example, study a phenomenon using a model that includes ions but not electrons, that model itself cannot tell us whether electron physics is important also on ion length and time scales. Ultimately, it is comparing model results to observations that must provide the answer to the question of model applicability, and it is the combined use of experiments and mathematical modelling that will advance space science in the future.

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