Electrostatic fluctuations in plasmas with distribution functions described by simple pole expansions

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The spectral densities of electrostatic fluctuations in plasmas where both the ion and electron distribution are described by simple pole expansions are calculated, for both one-dimensional and isotropic three-dimensional plasmas. Examples of electric field and density fluctuations are presented.

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I. INTRODUCTION

Electrostatic fluctuations appear in plasmas due to the discreteness of the plasma particles. A method for calculation of the spectral density of such fluctuations by superposition of dressed test particles was developed in the early 1960s[1,2]. This approach has been extended to treat electromagnetic fluctuations in order to study scattering of microwaves from such fluctuations in fusion plasma experiments[3]. Density fluctuations in the ionosphere have been measured by scattering of microwaves from ground-based radars[4]. However, the observed fluctuations are caused by turbulence rather than by particle discreteness. The thermal fluctuations are likely to be much smaller than the turbulent fluctuations.

Mace, Hellberg, and Treumann used the dressed test particle method to calculate the spectral density for electrostatic fluctuations in an isotropic three-dimensional plasma where the particles followed a kappa distribution[5]. They showed that the presence of super-thermal particles can significantly alter the excitation of the fluctuations. In the presence of super-thermals the Debye length is smaller than the Debye length of a Maxwellian plasma, and hence the plasma parameter $g = 1/(n\lambda_D^3)$ is larger in the presence of super-thermals. Effects that are due to particle discreteness increase with g, and hence such effects, like fluctuations, will be more important in plasmas with super-thermal particles, where there are fewer particles in the Debye sphere compared with Maxwellian plasmas.

In this paper we will use simple pole expansions to study electrostatic fluctuations in plasmas. The simple pole expansion[6] of the distribution function is a sum of simple poles in the complex phase velocity plane

$$f(v) = \sum_i \frac{a_i}{v - b_i}. \qquad (1)$$

The integral along the real axis in the dispersion relation reduces to a sum of the residues a_i at the poles b_i, and finding the dispersion relations amounts to solving $\epsilon(k, \omega) = 0$, where the dielectric function $\epsilon(k, \omega)$ is given by Eq. (12) below. This method has been used to study dispersion relations for electron[6] and ion[7] waves. It has been shown that the simple pole expansion of the distribution function is equivalent to a Padé approximation of the plasma dispersion function[8]. The Debye length for particle species α with an even distribution function $f_\alpha(v)$, described by a simple pole distribution is[7]

$$\lambda_{D\alpha} = \left(-\omega_{p\alpha}^2 2\pi i \sum_{b_{i,\alpha} \in U} \frac{a_{i,\alpha}}{b_{i,\alpha}^2} \right)^{-1/2}, \qquad (2)$$

where U denotes the upper half plane. The total Debye length is given by $\lambda_D = \left(\sum_\alpha \lambda_{D\alpha}^{-2}\right)^{-1/2}$.

In this paper we will use simple pole distributions to study electrostatic fluctuations in plasmas. Specifically we will use distribution functions that can be written as sums of expansions of the Maxwellian[6]

$$\begin{aligned} f(x) &= M(x), \qquad x \equiv (v - v_d)/v_t \qquad (3)\\ M(x) &= \left[1 + \frac{x^2}{2} + \ldots + \frac{1}{m!}\left(\frac{x^2}{2}\right)^m \right]^{-1}, \end{aligned}$$

where M is 1 over a truncated Taylor expansion of $e^{x^2/2}$, v_d is the average drift velocity and v_t is the standard deviation of the limiting Maxwellian. The number of poles in the upper half plane is m, which does not denote the particle mass. As m tends to infinity M approaches the Maxwellian. For finite values of m the simple pole expansion has tails that are thicker than the Maxwellian tails, and the number of super-thermal particles increases with lower m.

This paper is organized as follows. In section II expressions for the spectral density of the electric field and the particle density are derived for isotropic three-dimensional distribution functions, where the one-dimensional projection of the distribution function is modelled by a simple pole expansion. To facilitate a comparison with measurements spectral densities integrated over wavenumber are also given, and numerical examples are presented. In section III the spectral densities are given for a one-dimensional plasma where the one-dimensional distribution function is modelled by a simple pole expansion. In section IV fluctuations in plasmas having two ion temperatures are studied. In section V the results are summarized.

II. ELECTROSTATIC FLUCTUATIONS IN AN ISOTROPIC THREE-DIMENSIONAL UNMAGNETIZED PLASMA

A. E-field

A detailed description of the calculation of the spectral density of electrostatic fluctuations in an isotropic three-dimensional plasma through superposition of the electric fields of dressed test particles can be found in the book by Krall and Trivelpiece[9], or in the paper by Mace, Hellberg, and Treumann[5]. Here only the result is stated, and it is shown how fluctuation spectral densities can be calculated for distribution functions modelled by simple pole expansions.

The spectral density $S(\mathbf{k},\omega)$ of the electric field fluctuations is a function, which integrated over all frequencies and wavenumbers yields the energy density $\frac{1}{2}\epsilon_0 \langle \mathbf{E}^2 \rangle$ of the fluctuating electric field. In a general three-dimensional plasma the relation between $S(\mathbf{k},\omega)$ and $\frac{1}{2}\epsilon_0 \langle \mathbf{E}^2 \rangle$ is

$$\frac{1}{2}\epsilon_0 \left\langle \mathbf{E}^2 \right\rangle = \iiiint S(\mathbf{k},\omega)\, \frac{d^3\mathbf{k}}{(2\pi)^3}\, \frac{d\omega}{2\pi}. \tag{4}$$

The spectral density of the electric field fluctuations is

$$S(\mathbf{k},\omega) = \sum_\alpha \frac{\pi n_\alpha q_\alpha^2}{\epsilon_0} \frac{F_\alpha^{(0)}(\omega/k)}{k^3 |\epsilon(\mathbf{k},\omega)|^2}, \tag{5}$$

where n_α is the density and q_α the charge of the particles of species α, $\epsilon(\mathbf{k},\omega)$ is the dielectric function. $F_\alpha^{(0)}(u)$, the one-dimensional projection of the distribution function on an axis parallel to $\mathbf{k}$, is defined by

$$F_\alpha^{(0)}(u) = \iiint \delta\left(u - \frac{\mathbf{k}\cdot\mathbf{v}'}{k}\right) f_\alpha^{(0)}(\mathbf{v}')\, d^3\mathbf{v}' \tag{6}$$

The dielectric function $\epsilon(\mathbf{k},\omega)$ is given by

$$\epsilon(\mathbf{k},\omega) = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \iiint \frac{\mathbf{k}\cdot\partial f_\alpha^{(0)}(\mathbf{v})/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}}\, d^3\mathbf{v} \tag{7}$$

Both $\mathbf{k}$ and ω are real-valued quantities. $\omega = \mathbf{k}\cdot\mathbf{v}'$ is the frequency of resonance between a dressed test particle moving with velocity $\mathbf{v}'$ and a wave with wavenumber $\mathbf{k}$.

Since the $\epsilon(\mathbf{k},\omega)$ is dependent only on the component of the particle velocity that is parallel to $\mathbf{k}$ we can integrate over the perpendicular components. For the isotropic three-dimensional plasma the spectral density and the dielectric function will be simplified to functions of the scalar variables ω and $k = |\mathbf{k}|$, and are given by

$$S(k,\omega) = \sum_\alpha \frac{\pi n_\alpha q_\alpha^2}{\epsilon_0} \frac{F_\alpha^{(0)}(\omega/k)}{k^3 |\epsilon(k,\omega)|^2}. \tag{8}$$

and

$$\epsilon(k,\omega) = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \int \frac{k dF_\alpha^{(0)}(u)/du}{\omega - ku}\, du \tag{9}$$

instead of equations (5) and (7). Equation (9) follows from Eq. (7) and the fact that $\mathbf{k}\cdot\partial f_\alpha^{(0)}(\mathbf{v})/\partial\mathbf{v}$ is k times the derivative of $f_\alpha^{(0)}(\mathbf{v})$ in the direction in velocity space given by $\mathbf{k}$, and $\mathbf{k}\cdot\mathbf{v}$ in the denominator is k times the velocity component in that direction[9]. In the isotropic three-dimensional plasma the energy density is then given by the integral

$$\frac{1}{2}\epsilon_0 \left\langle \mathbf{E}^2 \right\rangle = \frac{1}{4\pi^3} \iint k^2 S(k,\omega)\, dk\, d\omega. \tag{10}$$

We will use simple pole expansions of the one-dimensional distribution function

$$F_\alpha^{(0)}(\omega/k) = \sum_i \frac{a_{i,\alpha}}{\omega/k - b_{i,\alpha}}. \tag{11}$$

Previously[6–8] the simple pole distributions have been used to find dispersion relations by assuming a real-valued wavenumber k, and solving $\epsilon(k,\omega) = 0$ for solutions with a complex ω. Here both k and ω are real and instead of solving $\epsilon(k,\omega) = 0$ we calculate the value of $\epsilon(k,\omega)$. As in the previous work this is done by closing the path of integration in the upper half plane, and using the residue theorem. The dielectric function is then

$$\epsilon(k,\omega) = 1 - 2\pi i \sum_\alpha \omega_{p\alpha}^2 \sum_{b_{i,\alpha}\in U} \frac{a_{i,\alpha}}{(\omega - k b_{i,\alpha})^2}. \tag{12}$$

The spectral density $S(k,\omega)$ follows by inserting (11) and (12) in (8):

$$S(k,\omega) = \frac{\sum_\alpha \frac{\pi n_\alpha q_\alpha^2}{\epsilon_0} \sum_i \frac{a_{i,\alpha}}{\omega/k - b_{i,\alpha}}}{k^3 \left|1 - 2\pi i \sum_\alpha \omega_{p\alpha}^2 \sum_{b_{i,\alpha}\in U} \frac{a_{i,\alpha}}{(\omega - k b_{i,\alpha})^2}\right|^2}. \tag{13}$$

To be able to compare the theory to measurements at one point in space the spectral density is integrated over all wavenumbers giving the energy density as a function of frequency only:

$$\begin{aligned} P_{EE}(\omega) &= \iiint S(\mathbf{k},\omega)\, \frac{d^3\mathbf{k}}{(2\pi)^3} \qquad (14) \\ &= \frac{1}{2\pi^2} \int k^2 S(k,\omega)\, dk \\ &= \int \frac{\frac{1}{2\pi} \sum_\alpha \frac{n_\alpha q_\alpha^2}{\epsilon_0} \sum_i \frac{a_{i,\alpha}}{\omega/k - b_{i,\alpha}}}{k \left|1 - 2\pi i \sum_\alpha \omega_{p\alpha}^2 \sum_{b_{i,\alpha}\in U} \frac{a_{i,\alpha}}{(\omega - k b_{i,\alpha})^2}\right|^2}\, dk \end{aligned}$$

The logarithm of the spectral density $\ln\left(S(k,\omega)/(m_e v_{te}^2/\omega_{pe})\right)$ for a plasma where both the electron and ion distributions are modelled by $m = 2$ expansions is shown in Fig. 1. The ratio between the electron and ion temperatures $T_e/T_i = 100$ and the ratio between ion and electron masses $m_i/m_e = 1836$. As expected the result is similar to that obtained by Mace, Hellberg, and Treumann[5]. The Langmuir and ion acoustic branches of the dispersion relations are clearly

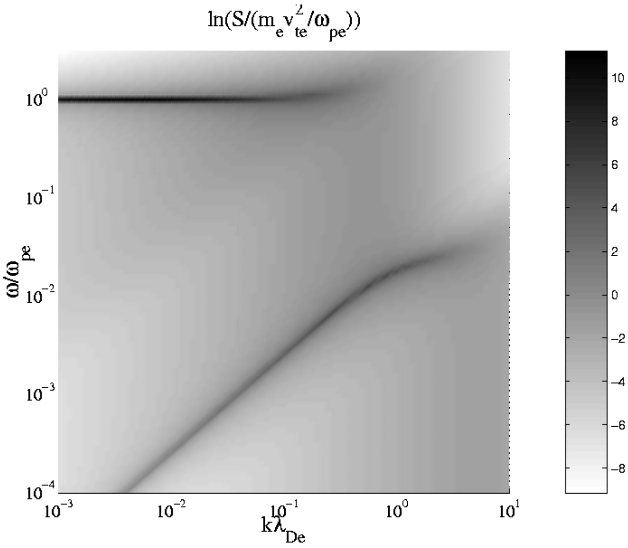

FIG. 1: The logarithm of the spectral density $\ln\left(S(k,\omega)/(m_e v_{te}^2/\omega_{pe})\right)$ for a plasma where both the electron and ion distributions are modelled by $m = 2$ expansions. The other parameters are $T_e/T_i = 100$ and $m_i/m_e = 1836$. The peak at $\omega = \omega_{pe}$ is not resolved.

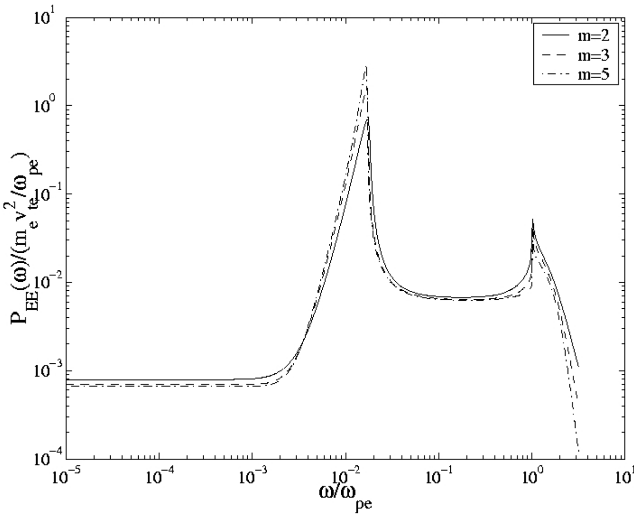

FIG. 2: $P_{EE}(\omega)$ is the integral with respect to k of $S(k,\omega)$ for a plasma where both the electron and ion distributions are modelled by $m = 2$ expansions (solid line), $m = 3$ expansions (dashed line), and $m = 5$ expansions (dash-dotted line). The other parameters are $T_e/T_i = 100$ and $m_i/m_e = 1836$. The peak at $\omega = \omega_{pe}$ is not resolved. The spectral density is integrated from $k\lambda_{De} = 0.001$ to $k\lambda_{De} = 1$.

seen in Fig. 1. This is because $|\epsilon(k,\omega)|^2$ appears in the denominator of Eq. (8), and although $\epsilon(k,\omega)$ never is equal to zero in a stable plasma it is small in the neighbourhood of the normal modes. The excitation of fluctuation depends both on the dispersion relation, and on the availability of particles at phase velocities close to the normal modes. That is to say that fluctuations appear if the distribution function $F(\omega/k)$ is not small for such k and ω where $\epsilon(k,\omega)$ is. The narrow peak in the fluctuation spectrum at the electron plasma frequency is not resolved in Fig. 1. The peak is so narrow that it falls between the grid points where $S(k,\omega)$ is calculated.

After numerical integration with respect to k from $k\lambda_{De} = 0.001$ to $k\lambda_{De} = 1$ the resulting $P_{EE}(\omega)$ is shown in Fig. 2. The integral is not evaluated from $k = 0$ to $k = \infty$ because in any real situation there are both lower and upper limits on k. In a laboratory experiment the lower limit on k is reached when the wavelength approaches the size of the experimental device. Any probe or optical measurement will constitute an average over a small volume, and the upper k limit is reached when the wavelength is comparable with the length of that volume. The narrow peak at the electron plasma frequency is not resolved in Fig. 2, and it is in reality much higher. The solid curve in Fig. 2 corresponds to a plasma with $m = 2$ distributions for both the electrons and the ions, for the dashed curve $m = 3$ and for the dash-dotted curve $m = 5$ for both electrons and ions. All other parameters are the same as for the plasma in Fig. 1. For a plasma with more super-thermal particles (lower m) the peak around ω_{pe} is broadened, since the excitation of fluctuations near the Langmuir branch increases due to the increased presence of resonant particles. Around the ion plasma frequency ($\omega_{pi} \approx 2 \times 10^{-2}\omega_{pe}$) the fluctuation level decreases with an increased number of super-thermals, because of the increased damping of the ion acoustic branch in this case.

B. Density

The relation between the perturbed density and the electric field is derived in appendix A. It has been assumed that the electrostatic potential is a slowly varying quantity ($\omega \ll \omega_{pe}$) so that the electrons have time to move, adjusting there density and maintaining quasi-neutrality. We are hence looking only at ion density fluctuations. For frequencies on the electron time scale the ions will not have time to move, and then the relevant density would be the electron density. For experimental purposes, the ion density is more interesting since it can be measured with probe or laser induced fluorescence techniques. Electron density measurements at frequencies on the order of the electron plasma frequency are extremely difficult to perform with wire probes. The relation between the perturbed density and the electric field is

$$n_1 = -i\frac{\epsilon_0}{e}\frac{1}{k}\left(k^2 + \frac{1}{\lambda_{De}^2}\right)E$$

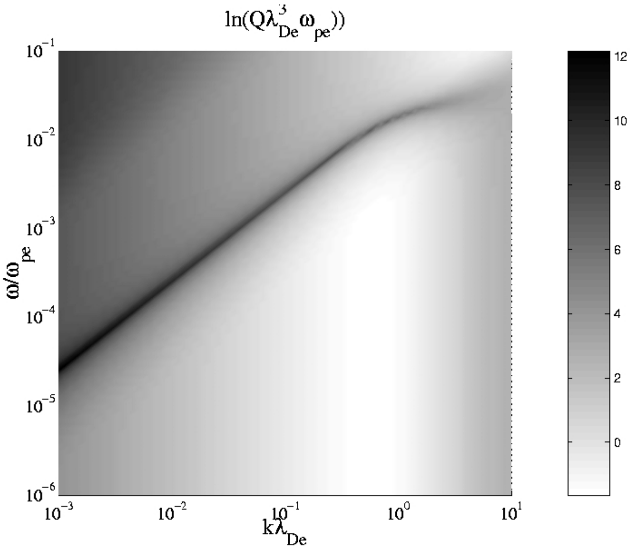

FIG. 3: The logarithm of the density spectral density $\ln\left(Q(k,\omega)\lambda_{De}^3\omega_{pe}\right)$. The plasma parameters are the same as in Fig. 1.

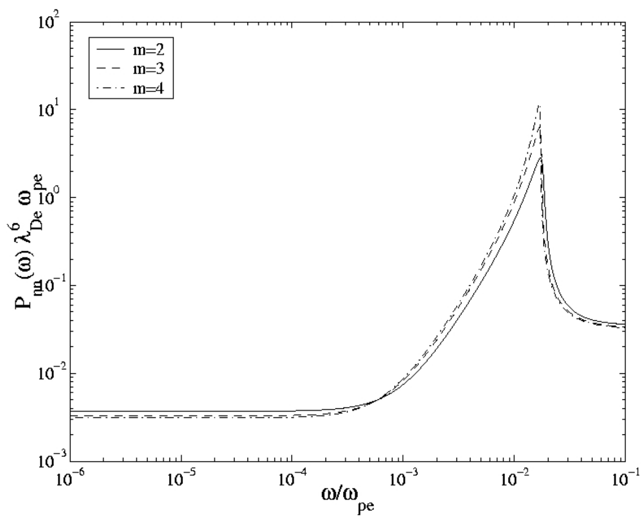

FIG. 4: $P_{nn}(\omega)$ is the integral of $Q(k,\omega)$ shown in Fig.3. The plasma parameters are the same as in figures 1 and 2. The solid curve shows $P_{nn}(\omega)$ for a plasma with $m=2$ expansions for both electrons and ions, the dashed curve $m=3$, and the dash-dotted curve $m=5$.

(see appendix A), and the average square of the density fluctuations is

$$\begin{aligned}\langle n_1^2\rangle &= \left\langle \frac{\epsilon_0^2}{e^2}\frac{1}{k^2}\left(k^2+\frac{1}{\lambda_{De}^2}\right)^2 E^2\right\rangle \\ &= \iiiint Q(\mathbf{k},\omega)\,\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{d\omega}{2\pi},\end{aligned} \tag{15}$$

where

$$Q(\mathbf{k},\omega) = \frac{2\epsilon_0}{e^2}\frac{1}{k^2}\left(k^2+\frac{1}{\lambda_{De}^2}\right)^2 S(\mathbf{k},\omega), \tag{16}$$

is the spectral density of the density fluctuations. After integration with respect to k we have the density fluctuation frequency spectrum $P_{nn}(\omega)$.

$$\begin{aligned}P_{nn}(\omega) &= \iiint Q(\mathbf{k},\omega)\,\frac{d^3\mathbf{k}}{(2\pi)^3} \\ &= \frac{1}{2\pi^2}\int k^2 Q(k,\omega)\,dk \\ &= \frac{\epsilon_0}{\pi^2 e^2}\int\left(k^2+\frac{1}{\lambda_{De}^2}\right)^2 S(k,\omega)\,dk\end{aligned} \tag{17}$$

The spectral density $Q(\omega,k)$ of the density fluctuations is shown in Fig. 3 and the frequency spectrum $P_{nn}(\omega)$ is shown in Fig. 4. The parameters are the same as for the E-field spectra shown in figures 1 and 2, i. e., $T_e/T_i = 100$, $m_i/m_e = 1836$, and both the electron and ion distributions are modelled by $m=2$ expansions (solid curve), $m=3$ (dashed curve), and $m=5$ (dash-dotted curve). As for the electric field fluctuations, described in section II A, the fluctuation levels near the ion acoustic branch are lower for plasmas with more super-thermals (lower m) due to increased damping.

III. FLUCTUATIONS IN A ONE-DIMENSIONAL PLASMA

The development of the theory of fluctuations in a one-dimensional plasma is analogous to that of a three-dimensional plasma. The differences between the one- and three-dimensional cases are that in the one-dimensional case all integrals over k- or x-space are one-dimensional, the density n is replaced by $n_\alpha^{(1D)}$ which is a line-density interpreted as the number of particles of kind α per metre. The particle charge q_α is replaced by a particle surface charge $\rho_{S,\alpha}$ in the one-dimensional theory.

A. E-field

In a one-dimensional plasma the energy of the electric field fluctuations $\frac{1}{2}\epsilon_0\langle E^2\rangle$ is given by an integral over all frequencies and wavenumbers of the one-dimensional spectral density $S^{(1D)}(k,\omega)$,

$$\frac{1}{2}\epsilon_0\left\langle E^2\right\rangle = \frac{1}{4\pi^2}\iint S^{(1D)}(k,\omega)\,dk\,d\omega. \tag{18}$$

The spectral density of the one-dimensional electric field fluctuations is given by

$$S^{(1D)}(k,\omega) = \sum_\alpha \frac{\pi n_\alpha^{(1D)}\rho_{S,\alpha}^2}{\epsilon_0}\frac{F_\alpha^{(0)}(\omega/k)}{k^3|\epsilon(k,\omega)|^2}, \tag{19}$$

where the dielectric function is

$$\epsilon(k,\omega) = 1 - 2\pi i\sum_\alpha \omega_{p\alpha}^2 \sum_{b_{i,\alpha}\in U}\frac{a_{i,\alpha}}{(\omega - k b_{i,\alpha})^2}.$$

as given by Eq. (12). The poles, $b_{i,\alpha}$, and the residues $a_{i,\alpha}$ are the poles and residues of the one dimensional distribution function $F_\alpha^{(0)}(v)$.

After integration of Eq. (19) over k the one dimensional frequency spectrum is obtained

$$P_{EE}^{(1D)}(\omega) = \int S^{(1D)}(k,\omega)\,\frac{dk}{2\pi} \qquad (20)$$
$$= \frac{1}{2}\int \frac{\sum_\alpha \frac{n_\alpha^{(1D)}\rho_{S,\alpha}^2}{\epsilon_0} \sum_i \frac{a_{i,\alpha}}{\omega/k - b_{i,\alpha}}}{k^3\left|1 - 2\pi i \sum_\alpha \omega_{p\alpha}^2 \sum_{b_{i,\alpha}\in U} \frac{a_{i,\alpha}}{(\omega - k b_{i,\alpha})^2}\right|^2}\,dk.$$

B. Density

From the relation between perturbed ion density and electric field (Eq. (A7)) we can calculate the average square of the density fluctuations for low frequencies ($\omega \ll \omega_{pe}$)

$$\left\langle \left(n_1^{(1D)}\right)^2 \right\rangle = \left\langle \frac{\epsilon_0^2}{e^2}\frac{1}{k^2}\left(k^2 + \frac{1}{\lambda_{De}^2}\right)^2 E^2 \right\rangle \qquad (21)$$
$$= \iint Q^{(1D)}(k,\omega)\,\frac{dk}{2\pi}\,\frac{d\omega}{2\pi},$$

where the spectral density of the one-dimensional density fluctuations is

$$Q^{(1D)}(k,\omega) = \frac{2\epsilon_0}{e^2}\frac{1}{k^2}\left(k^2 + \frac{1}{\lambda_{De}^2}\right)^2 S^{(1D)}(k,\omega). \qquad (22)$$

After integration with respect to k we have the one-dimensional density fluctuation frequency spectrum $P_{nn}^{(1D)}(\omega)$.

$$P_{nn}^{(1D)}(\omega) = \int Q^{(1D)}(k,\omega)\,\frac{dk}{2\pi} \qquad (23)$$
$$= \frac{\epsilon_0}{\pi e^2}\int \frac{1}{k^2}\left(k^2 + \frac{1}{\lambda_{De}^2}\right)^2 S^{(1D)}(k,\omega)\,dk$$

For three-dimensional isotropic distribution functions $S(k,\omega)$ and $Q(k,\omega)$ are numerically the same as in the corresponding one-dimensional case. Hence $S^{(1D)}(k,\omega)$ and $Q^{(1D)}(k,\omega)$ for a plasma with the same parameters as in figures 1 and 3 will look exactly as $S(k,\omega)$ and $Q(k,\omega)$ that are shown in those figures. The spectral densities integrated over k, $P_{EE}^{(1D)}(\omega)$ and $P_{nn}(\omega)$ are not the same in the one- and three-dimensional cases. For the parameters considered so far, i. e., $T_e/T_i = 100$, $m_i/m_e = 1836$, $P_{EE}^{(1D)}(\omega)$ is shown in Fig. 5 and $P_{nn}^{(1D)}(\omega)$ is shown in Fig. 6. The lower limit of integration is $k\lambda_{De} = 0.001$ and the upper is $k\lambda_{De} = 1$. In figures 5 and 6 the solid lines show a case where both the electron and ion distributions are modelled by $m = 2$ expansions. The dashed and dash-dotted lines show distributions modelled by $m = 3$ and $m = 5$ expansions respectively. Like the isotropic three-dimensional plasma the peak at ω_{pe} in the one-dimensional plasma is broader for lower values of m when there are more super-thermal particles, and the fluctuation levels at ion acoustic frequencies are lower for lower m-values.

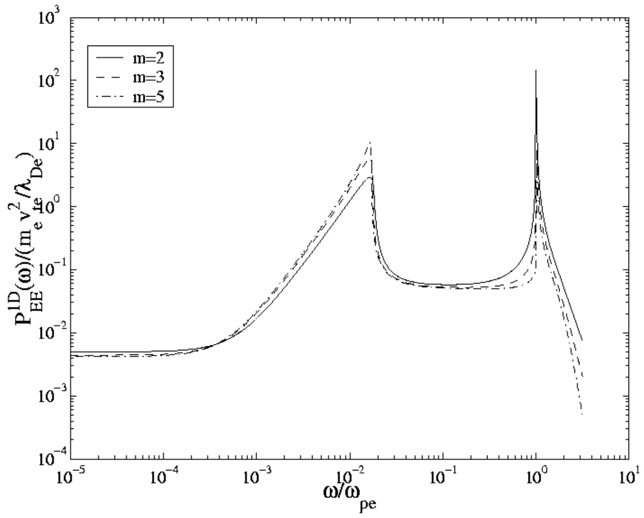

FIG. 5: $P_{EE}^{(1D)}(\omega)$ is the integral with respect to k of $S^{(1D)}(k,\omega)$ for a plasma where both the electron and ion distributions are modelled by $m = 2$ expansions (solid line), $m = 3$ expansions (dashed line), and $m = 5$ expansions (dash-dotted line). The other parameters are $T_e/T_i = 100$ and $m_i/m_e = 1836$. The peak at $\omega = \omega_{pe}$ is not resolved. The spectral density is integrated from $k\lambda_{De} = 0.001$ to $k\lambda_{De} = 1$.

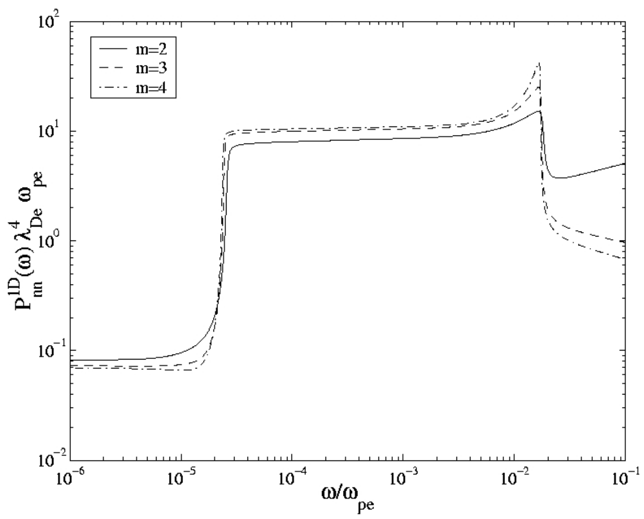

FIG. 6: $P_{nn}^{(1D)}(\omega)$ is the integral of $Q^{(1D)}(k,\omega)$ for a plasma where both the electron and ion distributions are modelled by $m = 2$ expansions (solid line), $m = 3$ expansions (dashed line), and $m = 5$ expansions (dash-dotted line). The other parameters are $T_e/T_i = 100$ and $m_i/m_e = 1836$. The peak at $\omega = \omega_{pe}$ is not resolved. The spectral density is integrated from $k\lambda_{De} = 0.001$ to $k\lambda_{De} = 1$.

In the three-dimensional case the slopes of the frequency spectra in the ion acoustic frequency range are steeper, by two orders of magnitude, than in the one-dimensional case. This applies to both the density and the electric field spectra. It is perhaps more evident in the density spectrum shown in Fig. 6, where the spectrum is nearly flat across the whole ion acoustic range. The reason for the difference in slope is the k^2 factor in the integrand of the three-dimensional frequency spectra (Eq. (14) and Eq. (17)) and not in the one-dimensional spectra (Eq. (20) and Eq. (23)). Hence relatively less weight is given to the low k region in the three-dimensional than in the one-dimensional plasma, and since ω is proportional to k for an acoustic wave, and the fluctuating energy concentrated near the normal modes, the fluctuation level rises faster with k in the three-dimensional case.

In the presence of a magnetic field the particle motion is restricted to the direction along the magnetic field lines. Hence the plasma in such cases could effectively be one-dimensional. A measurement of the slope of the fluctuation spectrum could then determine whether the fluctuations are one- or three dimensional, and hence whether assuming a one-dimensional plasma is a good approximation. One should keep in mind, however, that the present theory does not include any effects of the magnetic field other than the possible one-dimensionality, and that cyclotron resonances are not accounted for.

The sharp cutoff in $P_{nn}^{(1D)}(\omega)$ at $\omega \approx 3 \times 10^{-5}\omega_{pe}$ appears where the ion acoustic branch of the dispersion relation crosses the lower limit of integration $k\lambda_{De} = 0.001$. Similarly the sharp decrease in fluctuation levels above the ω_{pi} is influenced by the upper limit of integration $k\lambda_{De} = 1$.

IV. FLUCTUATIONS FOR DISTRIBUTIONS WITH TWO ION TEMPERATURES

Weakly damped acoustic-like modes with a phase speed lower than the ion sound speed can exist in plasmas where the ion distribution consists of two components that have different temperatures.[7] The quantity that is important for the dispersion relations is the thermal speed of the components, and similar modes occur in plasmas with two ion species.[10] In plasmas with two electron temperatures electron acoustic waves appear for the same reason.[11–13] Since high level fluctuations occur in the vicinity of the normal modes these acoustic-like modes affect the fluctuation spectra of plasmas with two ion temperatures.

In Fig. 7 the spectral density $S(k,\omega)$ of the fluctuating electric field is shown for two cases of plasmas with two-temperature ion distributions. The electrons are modelled by an $m = 3$ expansion and the ion to electron mass ratio is 1836. The ion distribution is composed of a cold component with a temperature $T_{i,c} = 0.01T_e$ and a hot component with a temperature $T_{i,h} = 0.16T_e$. The parameters are shown in table I which also includes a case with $n_h/n_i = 0.35$. In the upper panel of Fig.

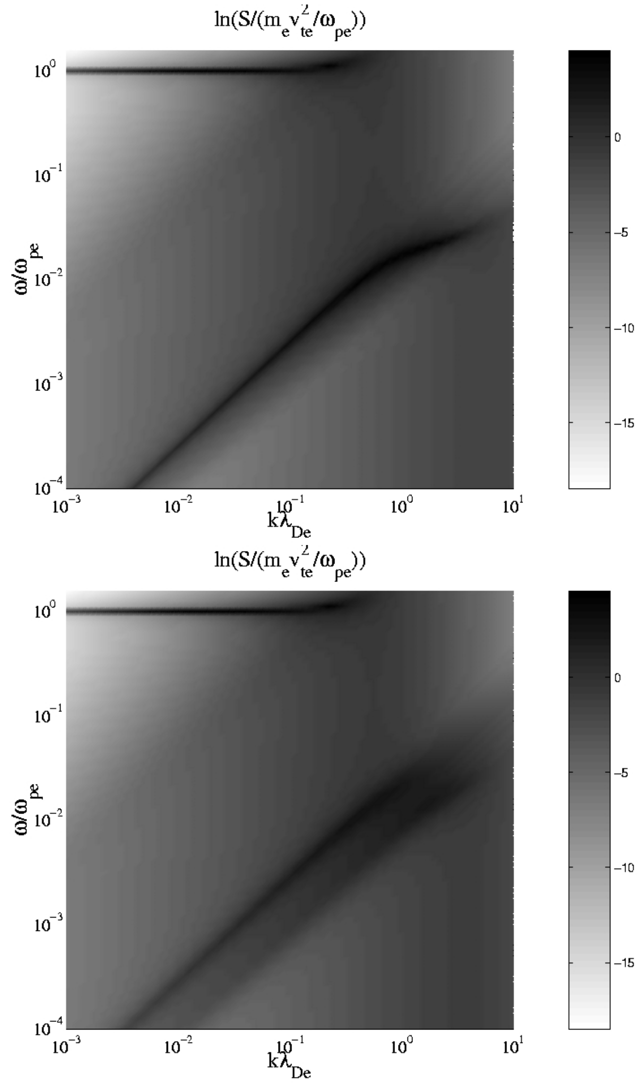

FIG. 7: Electric field fluctuations in a two-ion-temperature plasma. The upper panel shows the case where $n_h/n_i = 0.1$, and in the lower panel $n_h/n_i = 0.6$. Other parameters are shown in table I. The ion acoustic branch is broader than in the case of a one-component distribution, and in the lower panel where there are more hot ions the two ion modes can be seen. The peak at ω_{pe} is not resolved.

TABLE I: Parameters of the distribution functions for the plasmas that have their fluctuations shown in Fig. 7 and 9. The notation m_{i1}, m_{i2}, and m_e refers to the number of terms in the expansion and not the particle masses.

$\omega_{pi,1}^2$	$v_{ti,1}$	m_{i1}	$\omega_{pi,2}^2$	$v_{ti,2}$	m_{i2}	ω_{pe}^2	v_{te}	m_e
$0.9\omega_{pi}^2$	$0.1c_s$	5	$0.1\omega_{pi}^2$	$0.4c_s$	3	$1836\omega_{pe}^2$	$\sqrt{1836}c_s$	3
$0.65\omega_{pi}^2$	$0.1c_s$	5	$0.35\omega_{pi}^2$	$0.4c_s$	3	$1836\omega_{pe}^2$	$\sqrt{1836}c_s$	3
$0.4\omega_{pi}^2$	$0.1c_s$	5	$0.6\omega_{pi}^2$	$0.4c_s$	3	$1836\omega_{pe}^2$	$\sqrt{1836}c_s$	3

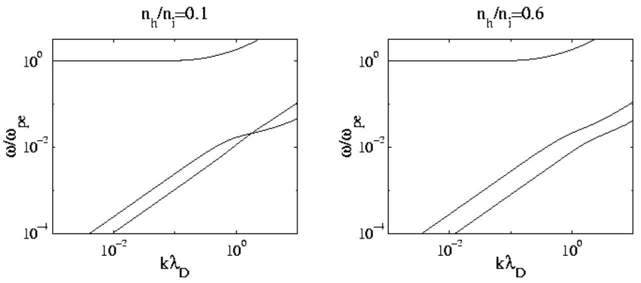

FIG. 8: Dispersion relations, for the three least damped modes, i. e. the Langmuir wave, the classical ion acoustic wave, and the slow acoustic-like wave. The left panel shows the $n_h/n_i = 0.1$ case and the right panel the $n_h/n_i = 0.6$ case.

7 $n_h/n_i = 0.1$, i. e. 10% of the ions belong to the hot distribution. The ion acoustic branch of the fluctuation spectral density is broader than in the case of a one-component ion distribution. In this case the slow waves are heavily damped.[7] The lower panel of Fig. 7 shows a case where $n_h/n_i = 0.6$. Here the ion acoustic branch is even wider than in the upper panel, and the two modes are seen more clearly. The dispersion relations for the three least damped modes are shown in Fig. 8 for the same two cases that are shown in Fig. 7. The Langmuir wave is in the upper part of the figures. The slow acoustic-like wave is the lower of the two parallel curves in the lower part of the figures. The dispersion relations shown are the real part of ω as a function of k calculated according to Löfgren and Gunell[6] for the Langmuir branch, and according to Gunell and Skiff[7] for the two ion modes. The one-dimensional density fluctuation frequency spectrum $P_{nn}^{(1D)}(\omega)$ is shown in Fig. 9 for $n_h/n_i = 0.1$ (solid curve), $n_h/n_i = 0.35$, (dashed curve), and $n_h/n_i = 0.6$ (dash-dotted curve). The fluctuation levels for the ion acoustic frequencies are lower in the cases with a higher relative hot ion density, because of the increased damping.[7] The cutoff at $\omega \approx 2 \times 10^{-5}\omega_{pe}$, that occurs when the ion acoustic mode enter the interval of integration at low k becomes less abrupt with increasing n_h/n_i because the spectral density is broader in k and ω due to the presence of the two modes.

The two-component distribution functions studied in this paper have both components centred at the same velocity. If one component is given a non-zero centre velocity, for example the hot low density component of the two-temperature distributions in this section, the damping rate can be reduced for the ion acoustic and ion acoustic-like waves. This will affect the fluctuations leading to more intense fluctuations in the ion acoustic frequency range.

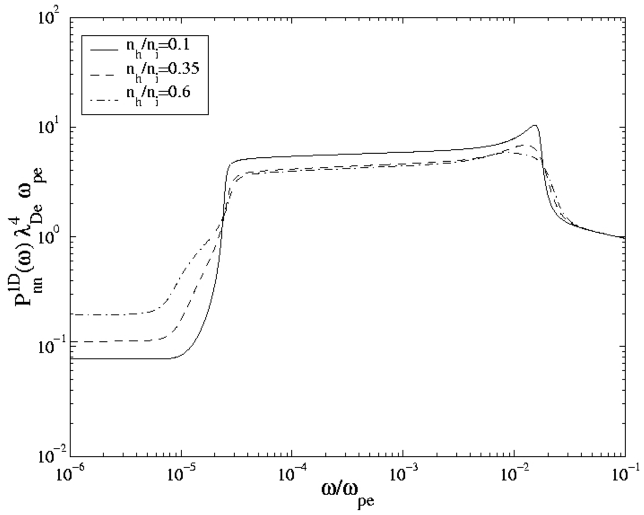

FIG. 9: One-dimensional density fluctuations integrated from $k = 0.001$ to $k = 1$ for three different cases, two of which (solid and dash-dotted lines) have their spectral density $S^{(1D)}(k, \omega)$ shown in Fig. 7. The dashed line shows a case where $n_h/n_i = 0.35$. Other parameters are shown in table I.

V. SUMMARY AND CONCLUSIONS

Electrostatic fluctuations occur in plasmas due to the discreteness of the plasma particles. Most of the energy of the fluctuations can be found in the vicinity of the normal modes in phase velocity space. Fluctuations further away from the normal modes are heavily damped and cannot propagate in the plasma. The particle distribution functions affect the normal modes and the damping of these modes. The distribution functions also affect the fluctuation spectrum in a more direct way, through the particle density in velocity space close to the normal modes. To have a high level of fluctuations in a region of velocity space there must be particles present that can excite a normal mode.

In this paper it is shown how the spectral density of these fluctuations can be calculated for non-Maxwellian plasmas whose distribution functions can be modelled by a simple pole distribution. This is done for one-dimensional and for isotropic three-dimensional plasmas. In the three-dimensional case a simple pole expansion is used to model the one-dimensional projection of the isotropic three-dimensional distribution function on an axis parallel to $\mathbf{k}$. The results for distributions with super-thermal particles are in agreement with the results obtained for kappa distributions,[5] that, for low κ-values also contain super-thermals.

For comparison with measurements made at one point in space it is interesting to integrate with respect to wavenumber and calculate the spectrum as a function of frequency. It is found that the slope of the frequency spectrum in the ion acoustic frequency range differs between the 1D and 3D cases by two orders of magnitude, being steeper in the three-dimensional case. This observation could be used to determine whether the fluctuations are predominantly one- or three-dimensional in plasma experiments where a magnetic field restricts the particle motion to one dimension, forcing the plasma to behave as an unmagnetized one-dimensional plasma. Effects of a magnetic fields other than the one-dimensionality are not taken into account in this theory.

In plasmas where the ion distribution function is composed of more than one component, so that it can be written as a sum of two or more simple pole expansions, acoustic-like weakly damped modes can appear on the ion acoustic time scale[7]. This modes also affect the fluctuation spectrum, and for an example of a distribution with two components that have the same centre velocity but different temperatures it is seen as a broadening of the ion acoustic branch of the fluctuations. For some parameters the two distinct modes are seen in the spectral density. If the two components have different centre velocities the damping can be decreased[7]. This will affect the fluctuation spectra, and such fluctuations can be studied with the method described here. Likewise the simple pole distributions are suitable for the study of electron acoustic waves and fluctuations that appear in plasmas with two electron temperatures.

Both the electron and the ion distributions influence the fluctuations even at ion acoustic frequencies. Thus measurements of both the electron and ion distribution functions will be needed in order to compare experimental and theoretical results. With laser induced fluorescence techniques fluctuations resolved in both frequency and particle velocity can be measured, and hence a desired continuation of existing fluctuation calculation theories would be to address fluctuations of the perturbed distribution function $f^{(1)}(k,\omega,v)$.

Acknowledgments

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APPENDIX A: RELATIONSHIP BETWEEN DENSITY AND ELECTRIC FIELD

Although the relationship between the density and the electric field is well known we derive it here for a plasma with a simple pole distribution. We will assume that the plasma potential varies slowly, so that it is stationary on the electron time scale. Hence the perturbed density n_1 is the ion density and it is obtained correctly only for low frequencies ($\omega \ll \omega_{pe}$).

When a distribution function that is known at one point in space changes under the influence of conservative forces the distribution at other points can be obtained by a mapping procedure. The poles and residues of the electron distribution function at a point where the electrostatic potential is ϕ is[6]

$$\begin{cases} b_{i,e}(\phi) = \sqrt{b_{i,e}^2(0) + \frac{2e\phi}{m_e}}, \ \Im b_{i,e}(\phi) > 0 \\ a_{i,e}(\phi) = a_{i,e}(0) \cdot \frac{b_{i,e}(0)}{b_{i,e}(\phi)} \end{cases} . \tag{A1}$$

This assumes that no part of the distribution is lost out of the system, which is an assumption that holds automatically since we are Fourier-transforming in space considering an infinite system where f_0 is the same everywhere.

With the aid of Eq. (A1) the electron density can be mapped.

$$n_e = n_0 2\pi i \sum_{b_{i,e}\in U} a_{i,e} \left(1 + \frac{2e\phi}{m_e b_{i,e}^2}\right)^{-1/2} \tag{A2}$$

For small perturbations $|2e\phi/m_e| \ll |b_{i,e}|^2$ the density can be approximated by

$$n_e = n_0 2\pi i \sum_{b_{i,e}\in U} a_{i,e} \left(1 - \frac{e\phi}{m_e b_{i,e}^2}\right) \tag{A3}$$

Hence the charge density is

$$\rho = en_1 + en_0 - en_0 2\pi i \sum_{b_{i,e}\in U} a_{i,e} \left(1 - \frac{e\phi}{m_e b_{i,e}^2}\right) \tag{A4}$$

which together with Poisson's equation $\rho = \epsilon_0 k^2 \phi$ yields

$$\frac{\epsilon_0 k^2}{e}\phi = n_1 + n_0 - n_0 + n_0 \phi 2\pi i \sum_{b_{i,e}\in U} \frac{a_{i,e}}{b_{i,e}^2}. \tag{A5}$$

Observing that the Debye length is given by[7] $\lambda_{De}^{-2} = \omega_{pe}^2 2\pi i \sum_{b_{i,e}\in U} \frac{a_{i,e}}{b_{i,e}^2}$ we have

$$E = ik\phi = ik\frac{n_1 e}{\epsilon_0}\frac{1}{k^2 + 1/\lambda_{De}^2}. \tag{A6}$$

The density perturbation can thus be found from the electric field

$$n_1 = -i\frac{\epsilon_0}{e}\frac{1}{k}\left(k^2 + \frac{1}{\lambda_{De}^2}\right)E \tag{A7}$$

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